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Quantum and Classical Evolutions of a Nonautonomous Dynamical System—A Comparison

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The influence of moving boundaries on the stability of quantum Hamiltonian systems, in particular on the dynamics of quantum versions of the classical Pustilnikov model, is investigated (the latter consists of a masspoint bouncing above an oscillating plate under the influence of constant gravity.) It is shown that, in contrast to the classical Pustilnikov model, generic time-periodic boundary conditions (including the Dirichlet condition) on the quantum models do not allow unlimited energy gain ("speeding up") of these systems.

KEY WORDS: Schrödinger equation on nonstationary domains; dynamics of (quantum) Pustilnikov models.

1. POSING THE PROBLEM

The last years have seen an enormous progress in the understanding of classical dynamical systems. Hence, the natural question arose whether the richness of classical dynamical phenomena would have any counterpart in the dynamics of quantum mechanical "analogies." Among others, quantum systems under time-dependent external perturbations were seen as candidates for the appearance of *quantum chaos* and much effort has been invested in the understanding of quantum Hamiltonian systems with near-integrable classical counterparts. (The review[(1)] contains a detailed discussion and a vast list of references.) In particular, one-dimensional systems under time-periodic perturbations have been studied and the following picture of their dynamical structure has emerged:

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The "generic" quantum system $H(t) = H_0 + V(t)$ is marked by stability in the sense that energy expectations of $H(t=0)$ with respect to the evolution generated by the family $\{H(t)\}$ remain bounded for all times—regardless of the fact that the "analogue" classical evolution might be unbounded in the corresponding phase space. That feature seems rather contradictory to the fact that any classical dynamics has to be embedded in the quantum set-up and it calls for an explanation. A possible mechanism for the differences between the classical and quantum evolutions might be a "smearing out" effect in the quantum propagation caused by uncertainty—i.e., irregular oscillations of classical trajectories are leveled out in the quantum evolution and merely in some $\hbar \downarrow 0$—limit the irregular dynamics exists. Another explanation, which is (*cautiously*) supported in this article, concerns the fact that in some cases the *chosen* quantum mechanical representation of the model cannot entirely reflect the wide dynamical variety of the classical system. This opinion applies in particular for models defined on time-dependent domains (as studied in the sequel), where the choice of boundary conditions is of prior importance to the dynamics of the corresponding system. However, as for the time being there are no experimental verifications of any of the above claims, the following discussion is purely mathematical in its nature and consequences.

This article is concerned with the stability (in the above sense, see also Proposition 2.7) of "quantum analogies" to the classical model rigorously studied by L. D. Pustilnikov.[2] As the latter consists of a mass-point bouncing above a periodically oscillating plate under the influence of constant gravity, introduce as the (heuristic) quantum scenario the following Schrödinger equation:

$$i\,\partial_t \Phi(t, y) = (-\partial_y^2 + y)\,\Phi(t, y) \tag{1.1}$$

defined on $\Omega(a) = \{(t, y) \in R^2 : y \in (a(t), \infty), t \in R\}$. Here the real-valued function $a \in C^3(R)$ models the boundary movement and fulfills $\dot{a}(t) := da/dt(t) \neq 0$ almost everywhere as well as $a(t) = a(t + k\Gamma)$, $k \in \mathrm{Z}$ with some period $\Gamma > 0$. To obtain a first impression of the influence of the "wall oscillations," apply the (point-wise) unitary shifts generated by $(t, y) \mapsto (t, x = y - a(t))$ and the (point-wise) unitary gauges $\mathfrak{Y}(t) = \exp(i\dot{a}(t)\,x/2)$ to map (1.1) onto $R \times R^+ \setminus \{0\}$. The result is

$$i\,\partial_t \Psi(t, x) = (-\partial_x^2 + x + \ddot{a}(t)\,x/2 - \dot{a}^2(t)/4 + a(t))\,\Psi(t, x) \tag{1.2}$$

To ensure unitary time evolutions $\mathfrak{U}(t, s)\,\psi(s)$ of initial states $\psi(s) \in L^2(R^+)$, the bracketed expressions in (1.2) have to be self-adjoint operators on $L^2(R^+)$ for all $t \in R$. Evidently, symmetry is obtained by requiring

$$\left.\frac{\psi_x}{\psi}(t)\right|_{x=0} = \alpha(t) \in \mathrm{R}, \qquad \alpha \in C^1(R) \tag{1.3}$$

The expressions (1.1)–(1.3) show that the interactions between quantum system and moving boundary consist of a *kinematical* and a *dynamical* contribution:

(A) There is a *geometrical* effect from the (physical) displacements of the boundary, expressed by the linear potential with time-dependent angle in (1.2). As demonstrated in the sequel, the periodically varying geometry alone has little influence on the stability of the system, both in quantum and classical terms, and unlimited energy transfer from the moving wall to the system is prohibited (at least for $\max_{t \in R} |\ddot{a}(t)| < 2$).

(B) In the classical model (in Section 3 and ref. 2) the origin of unlimited energy gain lies in the recurrence relation for the ball velocities before and after the nth-collision with the moving wall rather than in the nature of the Hamilton function, which represents the system between collisions. For initial conditions from a set of non-zero Lebesgue measure an "almost random" distribution of collision times results and it causes the speeding-up of the ball.[2] Hence, there exists a *dynamical* effect aside from the pure geometrical displacement discussed in (A). Its quantum analogon is the form of the boundary function α in (1.3). Following text-books, the exclusive choice should be $\alpha = \infty$ everywhere, i.e., the translation-invariant *Dirichlet condition*. However, to the best of the author's knowledge, there are no experiments describing a quantum wave packet colliding with a moving, impenetrable boundary. (Whatever that is in quantum mechanics?) Therefore, the proper choice of the boundary function α is an open problem from the physical point of view. Yet, this shortcoming does not prohibit the construction of various quantum models to explore different dynamical scenarios.

The present article finds its mathematical background in a recent work of the author.[3] A cornerstone in the discussion of various self-adjoint realizations of (1.2) in terms of ref. 3 is the existence of a reference model with pure point propagator. That reference model need not have any physically meaningful characteristics, it is only required as a starting point for the perturbation techniques employed in ref. 3. Thus, the first part of Section 2 is devoted to the construction of a reference model and it turns out that the latter carries the geometric features mentioned in (A), but neglects the dynamical effect discussed in (B). Then, based on these findings, the stability properties of a family of systems (1.2) characterized by certain time-periodic boundary functions α (including $\alpha(t) = \infty$) are investigated. Finally, in Section 3, the classical set-up is reviewed and its dynamics is compared to the quantum evolutions.

2. QUANTUM PUSTILNIKOV MODELS

The main obstacle towards a straightforward solution of (1.2) is its definition on R^+, i.e., the presence of the boundary at $x = 0$ has to be taken into account. (Defined on R, the model would be a version of the *AC-Stark effect*, which has seen an explicit solution,[4] for instance.) To guarantee existence and uniqueness of solutions of (1.2), (1.3), an explicitly solvable reference model reminding of the AC-Stark problem will be introduced. Its construction starts from the fact that the closure of $\dot{p} := -i\, d/dx$ on $C_0^\infty(R^+)$ does not have self-adjoint extensions on $\mathscr{H}^+ := L^2(R^+)$, i.e., there are no unitary translations on $\mathscr{H}^+$. Yet, the notion of "momentum" is all important when comparing quantum and classical dynamics. To overcome this difficulty, the following auxiliary set-up is introduced. (See ref. 5 for another application.)

Define the Hilbert space $\mathscr{H}$ as the direct sum

$$\mathscr{H} = \mathscr{H}^+ \oplus \mathscr{H}^- =: L^2(R^+) \oplus L^2(R^+).$$

Then the generator P of unitary shifts on $\mathscr{H}$ can be introduced as

$$\mathsf{P} = \begin{pmatrix} -i\,\dfrac{\partial}{\partial x} & 0 \\ 0 & i\,\dfrac{\partial}{\partial y} \end{pmatrix}$$

$$\mathscr{D}(\mathsf{P}) = \left\{ \Phi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} : \psi^+, \psi^- \text{ absolutely continuous}, \right.$$

$$\left. \psi^+(x=0) = \psi^-(y=0) \right\} \tag{2.1}$$

The boundary condition in (2.1) provides symmetry and self-adjointness follows from the one-parameter unitary translations $\{\mathfrak{T}(\mu), \mu \in R\}$,

$$\mathfrak{T}(\mu)\, \Phi = \begin{pmatrix} \psi^+(\cdot - \mu) \\ \psi^-(\circ + \mu) \end{pmatrix}$$

where, in case of $\mu > 0$, the condition $\psi^+(\hat{x} - \mu) \equiv \psi^-(\mu - \hat{x})$ for all $\hat{x} \leqslant \mu$ ensures unitarity. (Analogously for $\mu < 0$.) Remark that $\mathfrak{T}(\mu)$ *does not* split into two unitary mappings since neither $-i\,\partial/\partial x$ nor $i\,\partial/\partial y$ are self-adjoint operators. Accordingly, introduce the Laplacian P^2 on $\mathscr{H}$ with domain

$$\mathscr{D}(\mathsf{P}^2)=\{\Phi\in\mathscr{H},\ \mathsf{P}^2\Psi\in\mathscr{H}:\psi^+(x=0)=\psi^-(y=0),\\ \psi_x^+(x=0)=-\psi_y^-(y=0)\}\tag{2.2}$$

and the "position operators"

$$\mathsf{Q}=\begin{pmatrix}x&0\\0&y\end{pmatrix},\qquad \hat{\mathsf{Q}}=\begin{pmatrix}x&0\\0&-y\end{pmatrix}\tag{2.3}$$

From (2.1)–(2.3) the *standard commutation relations* are immediately deduced:

$$[\mathsf{P}^2,\hat{\mathsf{Q}}]=-2i\mathsf{P},\qquad [\mathsf{P},\hat{\mathsf{Q}}]=-i\tag{2.4}$$

Before characterizing the reference model, some additional information:

Lemma 2.1. Define P,P^2 and Q by (2.1)–(2.3) and assume $|\ddot{a}(t)|<2$ for all $t\in[0,\Gamma]$. Then the operators $\mathsf{P}^2+\beta\mathsf{Q}$ and $\mathsf{P}^2+\beta\mathsf{Q}+\gamma\mathsf{P}$ are self-adjoint on $\mathscr{H}$ with common domain $\mathscr{D}(\mathsf{P}^2+\mathsf{Q})$, where

$$\beta=\begin{pmatrix}1+\ddot{a}(t)/2&0\\0&1-\ddot{a}(t)/2\end{pmatrix}\qquad\text{for all (fixed)}\quad t\in R\quad\text{and}\quad \gamma\in R.$$

Proof. It suffices to discuss $\mathsf{P}^2+\mathsf{Q}$, since

$$\exp[i\gamma\hat{\mathsf{Q}}/2]\{\mathsf{P}^2+\beta\mathsf{Q}+\gamma\mathsf{P}\}\exp[-i\gamma\hat{\mathsf{Q}}/2]=\mathsf{P}^2+\beta\mathsf{Q}+\gamma^2/4\tag{2.5}$$

and $\mathscr{D}(\mathsf{P}^2+\beta\mathsf{Q})=\mathscr{D}(\mathsf{P}^2+\mathsf{Q})$ for all β follows from the Sturm–Liouville properties of the matrix entries.[6] As the equation $[d^2/dx^2+(\pm i-x)]\,\Phi^\pm=0$ is solved by the Airy function $\mathscr{A}i(\pm i-x)$, see ref. 7, the symmetric operator $\overline{\{\mathsf{P}^2+\mathsf{Q}\}\restriction\mathscr{C}_0^\infty(R^+)}$, with the set $\mathscr{C}_0^\infty(R^+):=\{\Psi\in\mathscr{H}:\psi^+,\psi^-\in C_0^\infty(R^+)\}$, has deficiency indices $(2,2)$, i.e., there exists a two-parameter family of self-adjoint extensions.[6] Among the possible choices is $\mathsf{P}^2+\mathsf{Q}$. ∎

The above properties are basic to the dynamics of the reference model:

Theorem 2.2. Let $a\in C^2(R)$, non-negative such that $|\ddot{a}(t)|<2$ everywhere and $a(t+k\Gamma)=a(t)$ for all $k\in Z$ and some $\Gamma>0$. Then the unitaries $\mathfrak{U}(a,t,s)$, $t\geqslant s$, given by

$$\mathfrak{U}(a, t, s) = \exp\left[i \int_s^t (\dot{a}^2(\tau)/4 - a(\tau))\, d\tau \right] \exp[-i\Lambda(t, s)]$$
$$\times \exp[-i(\dot{a}(t) - \dot{a}(s))\, \hat{\mathsf{Q}}/2] \exp[-i(\mathsf{P}^2 + \mathsf{Q})(t-s)]$$
$$\times \exp[i(a(t) - a(s) - \dot{a}(s)(t-s))\, \mathsf{P}]$$

represent the unique propagator to the Schrödinger equation

$$i\,\partial_t \Psi(t) = \left(\mathsf{P}^2 + \mathsf{Q}\left[1 + \ddot{a}(t)/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] - \dot{a}^2(t)/4 + a(t) \right) \Psi(t)$$

defined on $\mathscr{D}(\mathsf{P}^2 + \mathsf{Q})$. Here

$$\dot{\Lambda}(t, s) = (\dot{a}(t) - \dot{a}(s))^2/4 + (\dot{a}(t) - \dot{a}(s))(t-s) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Proof. (a) The family $\{\mathfrak{U}(a, t, s), (t, s) \in R^2\}$ is a propagator in the sense of ref. 4: Unitarity follows from Lemma 2.2. The groupoid structure is provided by the Baker–Campbell–Hausdorff formula[(4)] and relies on (2.4). Strong continuity of the first three exponentials is obvious and after transforming the last item with $\exp[-i\beta\hat{\mathsf{Q}}/2]$, see (2.5), its strong continuity is apparent.

(b) The family $\{\mathbf{U}(a, t, s), (t, s) \in R^2\}$ solves the Schrödinger equation in the theorem: Following the lead of ref. 4 in case of the "AC-Stark propagator," the claim is deduced with the aid of (2.4) through direct computation. ∎

The next statement is a consequence of Theorem 2.3.

Proposition 2.3. Let $t = \Gamma$, $s = 0$ and $\dot{a}(0) = 0$ in Theorem 2.3. Then the Floquet operator $\mathfrak{U}(a, \Gamma, 0)$ to the given Schrödinger equation reads

$$\mathfrak{U}(a, \Gamma, 0) = \exp\left[i \int_s^\Gamma (\dot{a}^2(\tau)/4 - a(\tau))\, d\tau \right] \exp[-i(\mathsf{P}^2 + \mathsf{Q})\, \Gamma]$$
$$=: \delta(a) \exp[-i(\mathsf{P}^2 + \mathsf{Q})\, \Gamma]$$

and the spectrum $\sigma(\mathfrak{U}(a, \Gamma, 0))$ is given by

$$\sigma(\mathfrak{U}(a, \Gamma, 0)) = \overline{\{\delta(a) \exp[-i\Gamma\varepsilon_k], k \in \mathrm{N}\}}$$

where the eigenvalues ε_k are determined by the corresponding zeros of the Airy function $\mathscr{A}i$ obeying $d\mathscr{A}i/d\xi|_{\xi = -\varepsilon_k} = 0$.

Proof. The choice $\dot{a}(0)=0$ is no restriction and the form of $\mathfrak{U}(a, \Gamma, 0)$ follows from the Γ-periodicity of a. The spectral properties are implied by the domain conditions (2.2) and the shape of the Airy function $\mathscr{A}i$, see ref. 7. ▮

Thus, the one-period solutions of the initial value problem in Theorem 2.3 are the solutions to the stationary set-up times a phase containing the classical action of a masspoint oscillating at velocity $\dot{a}$ above zero level. The reason for that trivial dynamics lies in the absence of boundary contributions as expressed by $\mathscr{D}(\mathsf{P})$, $\mathscr{D}(\mathsf{P}^2)$, Theorem 2.3 and the form of the commutator relations (2.4). However, note that the Hamiltonian in Theorem 2.3 is the "correct" one with respect to the model (1.2). Hence, the reference model is indeed the *geometrical model* and the relevant question concerns its relationship to systems containing "significant" boundary conditions at $x=0$. The following class is considered.

Proposition 2.4. Define the family $\{\mathsf{H}(a, \alpha, t), (a, \alpha) \in C^3_\Gamma(R) \times \hat{C}^1_\Gamma(R),\ t \in [0, \Gamma]\}$ by

$$\mathsf{H}(a, \alpha, t)\, \Psi = (-\Delta(\alpha(t)) + \mathsf{Q} + \ddot{a}(t)\, \hat{\mathsf{Q}}/2)\, \Psi,$$

$$\mathscr{D}(\mathsf{H}(a, \alpha, t)) = \left\{ \Psi \in \mathscr{H},\ \mathsf{H}(a, \alpha, t)\, \Psi \in \mathscr{H} : \frac{\psi^+_x}{\psi^+}(t, x=0) \right.$$

$$\left. = \frac{\psi^-_y}{\psi^-}(t, y=0) = \alpha(t) \text{ a.e.} \right\}$$

where $C^k_\Gamma(R) := \{ f \in C^k(R) \colon f(t) = f(t+k\Gamma), k \in Z, \Gamma > 0, \dot{f} \neq 0 \text{ a.e.}\}$ and $\hat{C}^k_\Gamma(R) := \{ f \in C^k(R) \text{ piece-wise:} f(t) = f(t+k\Gamma), k \in Z, \Gamma > 0, \}$. If $a \in C^3_\Gamma(R)$ with $|\ddot{a}(t)| < 2$ for all $t \in [0, \Gamma]$, then there exists a propagator $\mathfrak{U}(a, \alpha; t, s)$ to $i\, \partial_t \Psi = \mathsf{H}(a, \alpha, t)\, \Psi$ for all $\alpha \in \hat{C}^1_\Gamma(R)$.

(The Dirichlet condition $\alpha = \infty$ falls into that category.) The proof of Proposition 2.5 is based on extended Hilbert space methods and is part of the more general discussion in ref. 3. Applied to the present problem, the findings in ref. 3 lead to the main statement.

Theorem 2.5. Define the Hamiltonians $\{\mathsf{H}(a, \alpha, t), (a, \alpha) \in C^3_\Gamma(R) \times \hat{C}^1_\Gamma(R), t \in [0, \Gamma]\}$ as in Proposition 2.5 with $|\ddot{a}(t)| < 2$ everywhere. Then the corresponding Floquet operator $\mathfrak{U}(a, \alpha; \Gamma, 0)$ is pure singular for all $(a, \alpha) \in C^3_\Gamma(R) \times \hat{C}^1_\Gamma(R)$.

Singular continuous spectrum can be excluded to the extent that there are no sequences $(a_j, \alpha_j) \to (a, \alpha)$ in $C^3_\Gamma(R) \times \hat{C}^1_\Gamma(R)$ with $\sigma_{ac}(\mathfrak{U}(a_j, \alpha_j; \Gamma, 0)) \neq \emptyset$. This condition is necessary in Theorem 3.6 of ref. 3 for the existence of singular continuous propagators. However, as Theorem 3.6 of ref. 3 (presumably) does not cover all mechanisms leading to $\sigma_{sc}(\mathfrak{U}(a, \alpha; \Gamma, 0)) = \emptyset$, there is a chance to encounter some singular continuous quasi-energy spectrum for certain choices of (a, α). In this sense, pure point Floquet operators to families $\{\mathsf{H}(a, \alpha, t)\}$ are the most probable cases:

Proposition 2.6. The quantum system represented by the pure point Floquet operator $\mathfrak{U}(a, \alpha; \Gamma, 0)$ to the family $\{\mathsf{H}(a, \alpha, t), (a, \alpha) \in C^3_\Gamma(R) \times \hat{C}^1_\Gamma(R), t \in [0, \Gamma]\}$ is stable, i.e.,

$$\sup_{k \in N} |\langle \mathfrak{U}(a, \alpha; k\Gamma, 0)\, \phi_0, \mathsf{H}(a, \alpha, t=0)\, \mathfrak{U}(a, \alpha; k\Gamma, 0)\, \phi_0 \rangle_{\mathscr{H}}| < \infty$$

for a total set of initial conditions ϕ_0.

Proof. The strong continuity of $\mathfrak{U}(a, \alpha; t, s)$ provides $\psi_j \in \mathscr{D}(\mathsf{H}(a, \alpha, t=0))$ for all eigenfunctions ψ_j to $\mathfrak{U}(a, \alpha; \Gamma, 0)$, cf. ref. 3 and $\mathsf{H}(a, \alpha, t=0) + c$ is non-negative. Thus

$$\begin{aligned} &\|(\mathsf{H}(a, \alpha, t=0) + c)^{1/2}\, \mathfrak{U}(a, \alpha; k\Gamma, 0)\, \phi_0\|_{\mathscr{H}} \\ &\quad = \left\| \sum_{j \in \mathscr{J}} \exp(i\lambda_j k\Gamma) \langle \phi_0, \psi_j \rangle (\mathsf{H}(a, \alpha, t=0) + c)^{1/2}\, \psi_j \right\|_{\mathscr{H}} \\ &\quad \leqslant \sum_{j \in \mathscr{J}} |\langle \phi_0, \psi_j \rangle|\, \|(\mathsf{H}(a, \alpha, t=0) + c)^{1/2}\, \psi_j\|_{\mathscr{H}} < \infty \end{aligned} \tag{2.6}$$

for all $\phi_0 \in \text{linspan}\{\psi_j, j \in \mathscr{J}\}$. ∎

3. QUANTUM AND CLASSICAL PUSTILNIKOV MODELS COMPARED

The classical Pustilnikov model discusses the dynamics of a ball bouncing above a periodically oscillating table under the influence of constant gravity. In the point-wise rest frame of the oscillating plate the classical system is described by

$$\eta = p^2 + q, \qquad p_n^{(i)} = -p_n^{(f)} \tag{3.1}$$

with $(\textit{momentum}, \textit{position}) = (p = \dot{q}/2, q \geqslant a(t))$. With the (point-wise) canonical transformations $(p, q) \mapsto (\mathbf{P}, \mathbf{Q} = q - a(t))$ and $(\mathbf{P}, \mathbf{Q}) \mapsto (\hat{\mathbf{P}} = \mathbf{P} - \dot{a}(t)/2,$

$\hat{\mathbf{Q}} = \mathbf{Q}$) the problem on the fixed half-line $\{\mathbf{Q} \geqslant 0\}$ is expressed in analogy to (1.2) as

$$\hat{\mathbf{H}}(t) = \hat{\mathbf{P}}^2 + \hat{\mathbf{Q}}(1 + \ddot{a}(t)/2) + a(t) - \dot{a}(t)/4 \tag{3.2}$$

$$\dot{\hat{\mathbf{Q}}}_n^{(i)} - \dot{a}(t_n) = -\dot{\hat{\mathbf{Q}}}_n^{(f)} + \dot{a}(t_n) \tag{3.3}$$

Integration of Hamilton's equations determined by (3.2) and (3.3) yield an iterative map

$$v' = 2(t' - t) - v + 2\dot{a}(t'), a(t') = a(t) + v(t') - (t' - t)^2 \tag{3.4}$$

(Here v and v' are the ball velocities right after the collisions with the wall at $a(t)$, respectively $a(t')$, cf. ref. 2.) The main finding in ref. 2 shows that speeding-up occurs already for max $|\ddot{a}(t)| < 2$ for a set of positive Lebesgue measure of initial data. Evidently, the iteration scheme (3.4) contains intricate relations, reflecting the fact that the position of the wall at the nth-collision depends on $a(t_{n-1})$. To simplify matters, a fixed position of the wall is frequently assumed.[8] Then $t' - t = v$ and (3.4) reduces to the *standard map*.[9] In terms of (3.2), (3.3) the simplification provides the system

$$\hat{\mathbf{H}}_{\mathbf{d}}(t) = \hat{\mathbf{P}}^2 + \hat{\mathbf{Q}}, \qquad \dot{\hat{\mathbf{Q}}}_n^{(i)} - \dot{a}(t_n) = -\dot{\hat{\mathbf{Q}}}_n^{(f)} + \dot{a}(t_n) \tag{3.5}$$

which carries dynamical features similar to the full model over a certain limited range of the parameters, cf. ref. 8. Hence, the simplified model (3.5) describes the dynamical effects of the ball—moving plate collisions, whereas it neglects the geometrical features.

In view of Section 2, however, the classical geometrical model is of importance. In physical terms, this model corresponds to a ball bouncing in constant gravity above an oscillating plate where the latter is "instantly stopped" at the moments of impact. The *classical geometrical model* is characterized by

$$\hat{\mathbf{H}}_{\mathbf{g}}(t) = \hat{\mathbf{P}}^2 + \hat{\mathbf{Q}}(1 + \ddot{a}(t)/2) + a(t) - \dot{a}^2(t)/4, \dot{\hat{\mathbf{Q}}}_n^{(i)} = -\dot{\hat{\mathbf{Q}}}_n^{(f)} \tag{3.6}$$

Its dynamical features follow with *Ehrenfest's theorem*[10] from the *linear* quantum case in Section 2, since the operators P and $\hat{\mathsf{Q}}$ are related to the classical momentum and position observables via expectation values. In particular, the classical evolutions over full periods are governed by the quantum Floquet operator $\mathfrak{U}(a, \Gamma, 0)$ from Proposition 2.4:

Corollary 3.1. Let $\mathsf{X} \equiv \mathsf{P}$, respectively $\mathsf{X} \equiv \hat{\mathsf{Q}}$ as defined in (2.1) and (2.3). Then the sequences $\{\langle \Psi_0, [\mathfrak{U}(a, \Gamma, 0)^*]^N \mathsf{X} [\mathfrak{U}(a, \Gamma, 0)]^N \Psi_0 \rangle_{\mathscr{H}}\}_{N \in \mathbb{N}}$ are bounded for all $\Psi_0 \in \mathscr{D}(\mathsf{P}^2 + \mathsf{Q})$.

Proof. With the aid of (2.4) a short computation provides on $\mathscr{D}(\mathsf{P}^2 + \mathsf{Q})$

$$\hat{\mathsf{Q}}(N\Gamma) := \mathfrak{U}(a, N\Gamma, 0)^* \, \hat{\mathsf{Q}} \mathfrak{U}(a, N\Gamma, 0) = \hat{\mathsf{Q}} + N\Gamma\{2\mathsf{P} - N\Gamma\} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.7}$$

$$\mathsf{P}(N\Gamma) := \mathfrak{U}(a, N\Gamma, 0)^* \, \mathsf{P} \mathfrak{U}(a, N\Gamma, 0) = \mathsf{P} - N\Gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.8}$$

The non-negativity requirements for

$$\langle (\mathfrak{U}(a, N\Gamma, 0) \, \Psi_0)^+, x(\mathfrak{U}(a, N\Gamma, 0) \, \Psi_0)^+ \rangle_{\mathscr{H}^+},$$

respectively

$$\langle (\mathfrak{U}(a, N\Gamma, 0) \, \Psi_0)^-, y(\mathfrak{U}(a, N\Gamma, 0) \, \Psi_0)^- \rangle_{\mathscr{H}^-},$$

determine upper bounds $N^{\pm}_{k,\,\max}$ on the number $N^{\pm}$ of periods Γ for the individual runs. To continue the evolutions for $N^{\pm} > N^{\pm}_{k,\,\max}$, the "initial momenta" have to be reversed according to (3.7) and (3.8):

$$\langle (\Psi_0)^+, (-id/dx + x/\hat{t})(\Psi_0)^+ \rangle_{\mathscr{H}^+} \mapsto \langle (\Psi_0)^+, (id/dx - x/\hat{t})(\Psi_0)^+ \rangle_{\mathscr{H}^+} \tag{3.9}$$

where $\hat{t}$ is obtained from

$$\langle (\Psi_0^+, x\Psi_0^+ \rangle_{\mathscr{H}^+} = -\hat{t}\{2\langle \Psi_0^+, (-id/dx) \, \Psi_0^+ \rangle_{\mathscr{H}^+} - \hat{t} \, \| \Psi_0^+ \|^2_{\mathscr{H}^+}\}$$

The boundedness of the $\mathsf{P}(N\Gamma)$-expectations is implied by (3.9) and periodicity. ∎

Hence, an immediate consequence of Propositions 2.4 and 2.6 is

Proposition 3.2. There exists no speeding-up in the sense of ref. 2 in the geometrical model defined in (3.6).

Therefore, in similarity to the quantum model the absence of non-trivial, i.e., physically correct, boundary conditions prohibits an unlimited energy transfer even in case of a non-stationary wall.

To summarize, it remains to relate the above findings to the operator models discussed in Section 2. As all of these (generic) models are characterized by limited energy gain from the interactions with the "oscillating" boundary, the question whether they indeed represent true quantum analogies to the full classical Pustilnikov model (3.2), (3.3) seems to be appropriate. One might argue that the unbounded orbits in the classical model correspond to resonances between the quantum system and the boundary motion which occur for "noncharacteristic" pairs (a, α). (Similar to the kicked rotor model[9] where in exceptional cases a (singular) continuous spectrum might not be excluded). On the other hand, Pustilnikov's findings show that unbounded energy growth is present for a whole class of wall motions and in each case there is an associated set of nonzero Lebesgue measure of initial conditions—hardly an exceptional phenomenon.

Remind that the unbounded trajectories essentially origin from the "irregular" distribution in time of the wall-particle collisions. Therefore, in the opinion of the author, the attempt to model a complete quantum analogon in form of a time-periodic system as in Section 2 cannot be successful. (Compare, however, the numerical study of the problem in ref. 11.) A possible remedy might be the introduction of boundary conditions randomly distributed in time. Such systems will be discussed somewhere else.

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